SECTION 'A'

Q. 1. Answer any *five* of the following :

(a) Let S be the set of all real numbers except –1. Define * on S by

$$a * b = a + b + ab$$

Is (S, *) a group ?

Find the solution of the equation

$$2 * x * 3 = 7 \text{ in S.}$$ 12

(b) If G is a group of real numbers under addition and N is the subgroup of G consisting of integers, prove that G/N is isomorphic to the group H of all complex numbers of absolute value 1 under multiplication. 12

(c) Examine the convergence of

$$\int_0^1 \frac{dx}{x^{1/2}(1-x)^{1/2}}$$ 12

(d) Prove that the function f defined by

$$f(x) = \begin{cases} 1, & \text{when x is rational} \\ -1, & \text{when x is irrational} \end{cases}$$

is nowhere continuous. 12

(e) Determine all bilinear transformations which map the half plane Im (z) ≥ 0 into the unit circle $|w| \leq 1$. 12

(f) Given the programme

Maximize $u = 5x + 2y$

subject to $x + 3y \leq 12$

$3x - 4y \leq 9$

$7x + 8y \leq 20$

$x, y \geq 0$

Write its dual in the standard form. 12

Q. 2. (a) (i) Let O (G) = 108. Show that there exists a normal subgroup or order 27 or 9.

(ii) Let G be the set of all those ordered pairs (a, b) of real numbers for which $a \neq 0$ and define in G, an operation $\otimes$ as follows :

$$(a, b) \otimes (c, d) = (ac, bc + d)$$

Examine whether G is a group w.r.t. the operation $\otimes$. If it is a group, is G abelian ? 10

(b) Show that

$$Z[\sqrt{2}] = \{a + \sqrt{2}\, b \mid a, b \in Z\}$$

is a Euclidean domain. 30

Q. 3. (a) A twice differentiable function f is such that f (a) = f (b) = 0 and f (c) > 0 for a < c < b. Prove that there is at least one value ξ, $a < \xi < b$ for which $f''(\xi) < 0$. 20

(b) Show that the function given by

$$f(x,y) = \begin{cases} \dfrac{x^3 + 2y^3}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0 &, (x, y) > (0, 0) \end{cases}$$

(i) is continuous at (0, 0).

(ii) possesses partial derivatives

$f_x(0, 0)$ and $f_y(0, 0)$. 20

(c) Find the volume of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Q. 4. (a) With the aid of residues, evaluate

$$\int_0^{\pi} \frac{\cos 2\theta}{1 - 2a\cos\theta + a^2}\, d\theta, \; -1 < a < 1$$ 15

(b) Prove that all the roots of $z^7 - 5z^3 + 12 = 0$ lie between the circles $|z| = 1$ and $|z| = 2$. 15

(c) Use the simplex method to solve the problem

Maximize $u = 2x + 3y$

subject to $-2x + 3y \leq 2$

$3x + 2y \leq 5$

$x, y \geq 0$ 30

SECTION 'B'

Q. 5. Answer any *five* of the following :

(a) Solve :

$$px(z - 2y^2) = (z - qy)(z - y^2 - 2x^3)$$ 12

(b) Solve :

$$\frac{\partial^3 z}{\partial x^3} - 4\frac{\partial^3 z}{\partial x^2 \partial y} + 4\frac{\partial^3 z}{\partial x \partial y^2} = 2\sin(3x + 2y)$$ 12

(c) Evaluate

$$I = \int_0^1 e^{-x^2}\, dx$$

by the Simpson's rule

$$\int_a^b f(x)\, dx \approx \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2)]$$

$$+ 4f(x_3) + \ldots + 2f(x_{2n-2}) + 4f(x_{2n-1}) + f(x_{2n})]$$

with

$2n = 10, \Delta x = 0.1, x_0 = 0, x_1 = 0.1, \ldots, x_{10} = 1.0$ 12

(d) (i) Given the number 59.625 in decimal system. Write its binary equivalent. 6

(ii) Given the number 3898 in decimal system. Write its equivalent in system base 8. 6

(e) Given points $A(0, 0)$ and $B(x_0, y_0)$ not in the same vertical, it is required to find a curve in the x – y plane joining A to B so that a particle starting from rest will traverse from A to B along this curve without friction in the shortest possible time. If y = y (x) is the required curve find the function f (x, y, z) such that the equation of motion can be written as

$$\frac{dx}{dt} = f(x, y(x), y'(x)).$$ 12

(f) A steady inviscid incompressible flow has a velocity field

$$u = fx,\ v = -fy,\ w = 0$$

where f is a constant. Derive an expression for the pressure field p {x, y, z} if the pressure

$$p\{0, 0, 0\} = p_0 \text{ and } \vec{g} = -g\,\vec{i}_z.$$ 12

Q. 6. (a) The deflection of a vibrating string of length l, is governed by the partial differential equation $u_{tt} = C^2 u_{xx}$. The ends of the string are fixed at $x = 0$ and l. The initial velocity is zero. The initial displacement is given by

$$u(x, 0) = \frac{x}{l},\ 0 < x < \frac{l}{2}$$

$$= \frac{1}{l}(l - x),\ \frac{l}{2} < x < l.$$

Find the deflection of the string at any instant of time. 30

(b) Find the surface passing through the parabolas $z = 0$, $y^2 = 4ax$ and $z = 1$, $y^2 = -4ax$ and satisfying the equation

$$x\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial z}{\partial x} = 0$$ 15

(c) Solve the equation

$$p^2 x + q^2 y = z,\ p = \frac{\partial z}{\partial x},\ q = \frac{\partial z}{\partial y}$$

by Charpit's method. 15

Q. 7. (a) If Q is a polynomial with simple roots $\alpha_1, \alpha_2, \dots \alpha n$ and if P is a polynomial of degree < n, show that

$$\frac{P(x)}{Q(x)} = \sum_{k=1}^{n} \frac{P(\alpha_k)}{Q'(\alpha_k)\,(x-\alpha_k)}.$$

Hence prove that there exists a unique polynomial of degree < n with given values c_k at the point α_k, k = 1, 2, ... n. 30

(b) Draw a programme outline and a flow chart and also write a programme in BASIC to enable solving the following system of 3 linear equations in 3 unknowns x_1, x_2 and x_3 :

$$C * X = D$$

with

$$C = (c_{ij})^3_{i,j=1},\ X = (x_j)^3_{j=1},\ D = (d_i)^3_{i=1}.$$ 30

Q. 8. (a) A particle of mass m is constrained to move on the surface of a cylinder. The particle is subjected to a force directed towards the origin and proportional to the distance of the particle from the origin. Construct the Hamiltonian and Hamilton's equations of motion. 30

(b) Liquid is contained between two parallel planes, the free surface is a circular cylinder of radius a whose axis is perpendicular to the planes. All the liquid within a concentric circular cylinder of radius b is suddenly annihilated; prove that if P be the pressure at the outer surface, the initial pressure at any point on the liquid, distant r from the centre is

$$P\,\frac{\log r - \log b}{\log a - \log b}$$ 30